Variable Selection in Regression using Maximal Correlation and Distance Correlation

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Variable Selection

- Recent improvements in data collection technologies give rise to complex regression problems where the number of candidate predictor variables explaining the response variable may be very large.

- In most of these regression problems the main task is to select the most influential predictors explaining the response, and removing the others from the model.

- These problems are usually referred to as variable selection problems in the statistical literature.

Subset Selection

Consider the linear regression model

$$Y = X\beta + \epsilon, \quad (1)$$

where Y is a vector of length n representing the response variable, X is an n by p matrix representing the predictor variables, β is a vector of length p containing regression coefficients, and ϵ is a vector of length n containing independent normal noise terms.

The essential goal in variable selection is to divide X into the set of active terms X_A and the set of inactive terms X_I.

Issues:

- Comparison Criterion for two candidates of X_A.
  - Akaike Information Criterion: $AIC = n\log\left(RSS/n\right) + 2p$
  - Bayesian Information Criterion: $BIC = n\log\left(RSS/n\right) + p\log n$
  - Computationally Intensive Comparison Criteria: k-Fold Cross-Validation, etc.
- Computational Method. If there are p candidate predictors, there are $2^p - 1$ possible candidates for X_A. Ex: When $p = 20$ $\rightarrow$ 1,048,575 possible models to check.
  - Stepwise Methods (Forward and Backward).
  - Branch-and Bounds, Leaps-and-Bounds.
  - Stagewise Methods.

Shrinkage Methods

The discrete nature of subset selection methods may lead to high variance in some situations.

Due to their continuous nature, *shrinkage methods* may provide an alternative to the subset selection methods.

- Ridge Regression (Hoerl and Kennard, 1970a,b)
- Lasso (Tibshirani, 1996)
- LARS (Efron *et. al.*, 2004)

Dependence Measures

In virtually any field of statistics, there is a need for measuring the dependence between random variables.

According to Rényi (1959), a measure of dependence should satisfy the following postulates.

Rényi (1959) Postulates for Measures of Dependence

A) $\delta(X, Y)$ is defined for every pair X, Y neither of which is constant with probability 1.
B) $\delta(X, Y) = \delta(Y, X)$.
C) $0 \leq \delta(X, Y) \leq 1$.
D) $\delta(X, Y) = 0$ if and only if X and Y are independent.
E) $\delta(X, Y) = 1$ if either $X = g(Y)$ or $Y = f(X)$, where $g(\cdot)$ and $f(\cdot)$ are Borel-measurable functions.
F) If the Borel-measurable functions $g(\cdot)$ and $f(\cdot)$ map the real axis in a one-to-one way to itself, then $\delta(f(X), g(Y)) = \delta(X, Y)$.
G) If the joint distribution of X and Y is normal, then $\delta(X, Y) = |R(X, Y)|$, where $R(X, Y)$ is the correlation coefficient of X and Y.

Maximal Correlation

The maximal correlation S between two random variables (X, Y) is defined as

$$S(X, Y) = \sup_{f,g} \rho(f(X), g(Y)),$$

where ρ denotes the classical correlation coefficient, and the supremum is taken over all functions of X and Y with finite and positive non-zero variance.

If there exist some f_0 and g_0 such that $S(X, Y) = \rho(f_0(X), g_0(Y))$, we say that the maximal correlation of X and Y can be attained.

Maximal Correlation satisfies all 7 postulates listed by Rényi (1959).

Product Moment Correlation satisfies B, C, and G only.

Gebelein (1941)
Rényi (1959)
Csáki and Fisher (1963)
Breiman and Friedman (1985)
Koyak (1987)
Sethuraman (1990)
Dembo et. al. (2001)
Bryc et. al. (2005)
Yenigun et. al. (2011)

Distance Correlation

Consider random vectors X in $\mathbb{R}^p$ and Y in $\mathbb{R}^q$. The characteristic functions of X and Y are denoted by f_X and f_Y, respectively, and the joint characteristic function of X and Y is $f_{X,Y}$.
The distance covariance between X and Y is

$$V^2(X,Y) = \|f_{X,Y}(t,s) - f_X(t)f_Y(s)\|^2. \tag{2}$$

See Szekely, Rizzo, Bakirov (2007) for the norm $\|\cdot\|$.

Similarly, the distance variance of X is

$$V^2(X) = \|f_{X,X}(t,s) - f_X(t)f_X(s)\|^2, \tag{3}$$

and the distance correlation between X and Y is

$$R^2(X,Y) = \begin{cases} \frac{V^2(X,Y)}{\sqrt{V^2(X)V^2(Y)}}, & V^2(X)V^2(Y) > 0 \\ 0, & V^2(X)V^2(Y) = 0 \end{cases}. \tag{4}$$

For all distributions with finite first moments, distance correlation R generalizes the idea of correlation in two fundamental ways:

1. $R(X, Y)$ is defined for X and Y in arbitrary dimensions,
2. $R(X, Y) = 0$ if and only if X and Y are independent.

Distance correlation satisfies the Rényi postulates A, B, C, D. The rest is partly satisfied. E is satisfied for linear functions, F is satisfied for orthogonal transformations. As for G, if X and Y are bivariate normal, R is a function of ρ.

Proposed Methods

We propose two model selection methods based on the dependence measures distance correlation and maximal correlation.

- Stepwise regression using distance correlation
- Stepwise regression using maximal correlation

Partial Distance (/Maximal) Correlation

We begin with defining partial distance (/maximal) correlation. Consider random variables X, Y, and a possibly vector valued random variable Z. Given Z, the partial distance (/maximal) correlation between X and Y is computed as follows:

- Regress X on Z, denote the error terms by R_X.
- Regress Y on Z, denote the error terms by R_Y.
- The distance (/maximal) correlation between R_X and R_Y is the partial distance correlation between X and Y, given Z.

Stepwise Regression Using Distance (/Maximal) Correlation

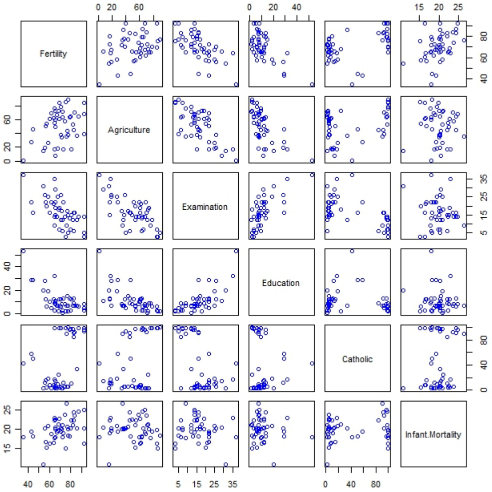
Then we can define a stepwise regression procedure, using distance (/maximal) correlation as follows:

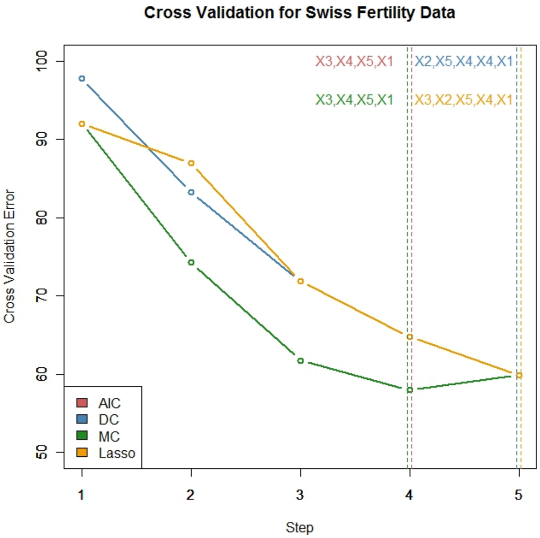
1. Consider all candidate predictor variables individually and find the one which has the largest distance (/maximal) correlation with the dependent variable.
2. For the remaining steps, add one more term such that the partial distance (/maximal) correlation with the dependent variable, given the previously entered variable(s), is largest.
3. Stop when all terms have entered the model. The step with the smallest cross-validation error is the selected model.

Illustration on Swiss Fertility Data

Standardized fertility measure and socio-economic indicators for each of 47 French-speaking provinces of Switzerland at about 1888.

- Y - Common standardized fertility measure (Fertility)
- X_1 - Percentage of males involved in agriculture as occupation (Agriculture)
- X_2 - Percentage of draftees receiving highest mark on army examination (Examination)
- X_3 - Percentage of education beyond primary school for draftees (Education)
- X_4 - Percentage of Catholic (Catholic)
- X_5 - Live births who live less than 1 year (Infant Mortality)


Fertility
Agriculture
Examination
Education
Catholic
Infant.Mortality


Cross Validation for Swiss Fertility Data
X3,X4,X5,X1
X2,X5,X4,X4,X1
X3,X4,X5,X1
X3,X2,X5,X4,X1
Cross Validation Error
50
60
70
80
90
100
1
2
3
4
5
Step
AIC
DC
MC
Lasso

Simulation Study

We consider 6 cases.

- Case 1: Linear Relations
- Case 2: Non-Linear Relations
- Case 3: Dependent but Uncorrelated Variables
- Case 4: Constant Collinearity Among Predictors
- Case 5: Toeplitz Collinearity Among Predictors
- Case 6: A Generalized Linear Model: Gamma Regression

For each case we considered $N = 100$ samples of size $n = 100$.

Case 1: Linear Relations

We consider a total of $p = 8$ candidate predictors having independent standard normal distributions, $q = 3$ of which are related with the dependent variable via:

$$Y = X\beta + \epsilon,$$

where $\beta = [1, 1, 1, 0, 0, 0, 0, 0]$ and $\epsilon \sim N(0, \sigma = 2)$.

Case 2: Non-Linear Relations

We consider a total of $p = 8$ candidate predictors from the following distributions: $X_1 \sim N(0, 1)$, $X_2 \sim N(0, 2)$, $X_3 \sim U(-1.5, 1.5)$, $X_4, ..., X_8 \sim U(-1, 1)$. The first $q = 4$ are related with the dependent variable via:

$$Y = \log[4 + \sin(3X_1) + \sin(X_2) + X_3^2 + X_4 + 0.1\epsilon],$$

where $\epsilon \sim N(0, \sigma = 1)$.

Case 3: Dependent but Uncorrelated Variables

We consider a total of $p = 8$ candidate predictors from the following distributions: $X_1 \sim N(0, 1.4)$, $X_2 \sim U(-1.7, 1.7)$, $X_3 \sim N(0, 0.8)$, $X_4, ..., X_8 \sim N(0, 1)$. Let us define $Y_1, ..., Y_3$ as follows:

$$Y_1 = |X_1|, \quad Y_2 = X_2^2, \quad Y_3 = X_3^2.$$

It can be shown that the pairs (X_i, Y_i), $i = 1, 2, 3$, are uncorrelated. We define the dependent variable as

$$Y = |X_1| + X_2^2 + X_3^2.$$

Case 4: Constant Collinearity Among Predictors

We consider a total of $p = 8$ candidate predictors from a multivariate normal distribution, $\mathbf{X} \sim N_P(\mathbf{0}, \Sigma)$, where

$$\Sigma = \begin{bmatrix} 1 & \theta & \cdots & \theta \\ \theta & 1 & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta & \theta & \cdots & 1 \end{bmatrix}.$$

We set $\theta = 0.6$. The first $q = 3$ of these variables are related with the dependent variable via:

$$Y = X\beta + \epsilon,$$

where $\beta = [1, 1, 1, 0, 0, 0, 0, 0]$ and $\epsilon \sim N(0, \sigma = 2)$.

Case 5: Toeplitz Type Collinearity Among Predictors

This is the same as Case 4, but

$$\Sigma = \begin{bmatrix} 1 & \theta & \theta^2 & \cdots & \theta^{p-1} \\ \theta & 1 & \theta & \cdots & \theta^{p-2} \\ \theta^2 & \theta & 1 & \cdots & \theta^{p-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta^{p-1} & \theta^{p-2} & \theta^{p-3} & \cdots & 1 \end{bmatrix}.$$

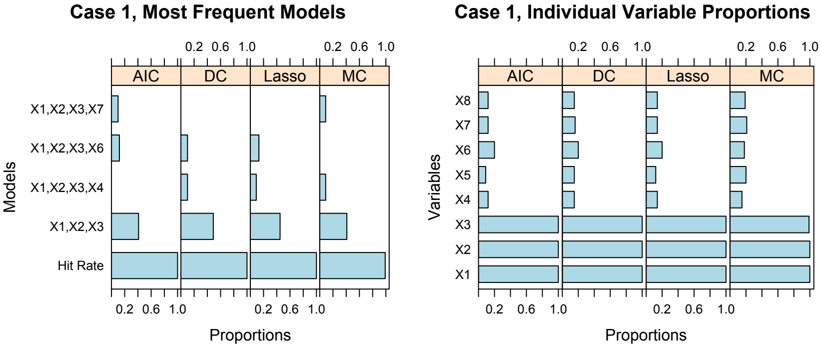
Case 6: A Generalized Linear Model (Gamma Regression)

We consider $p = 8$ candidate predictors following standard normal distribution, $q = 3$ of which are related with the response via:

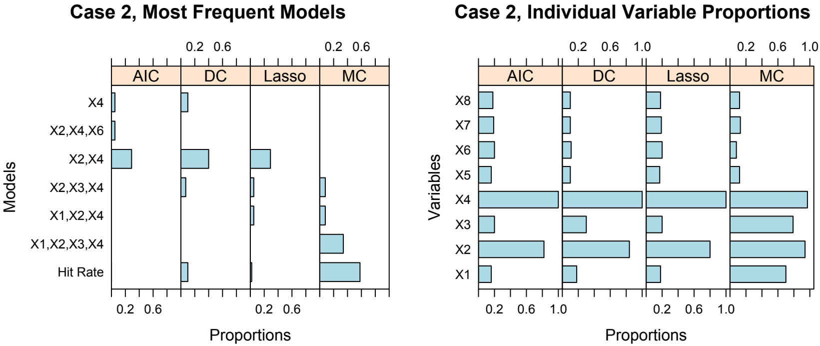
$$L = X\beta,$$

with $\beta = [0.25, 0.25, 0.25, 0, 0, 0, 0, 0]$. The link function is the log function, thus the mean vector of the responses are $\hat{\mu} = e^L$. Responses are generated from gamma distribution with mean $\hat{\mu}$ and unit variance

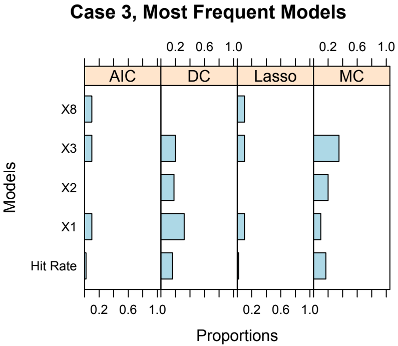
Case 1: Linear Relations

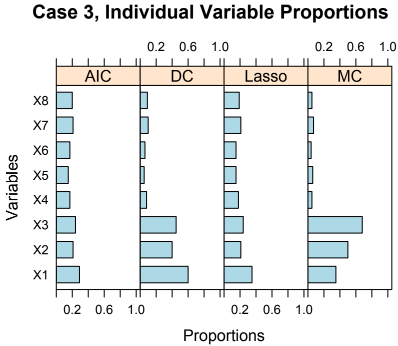


Case 2: Non-Linear Relations

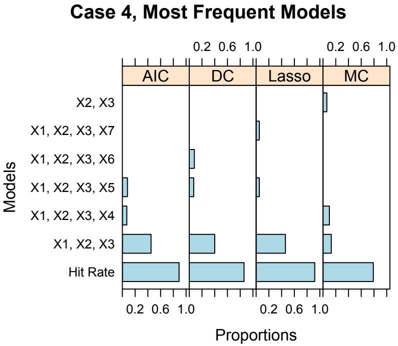


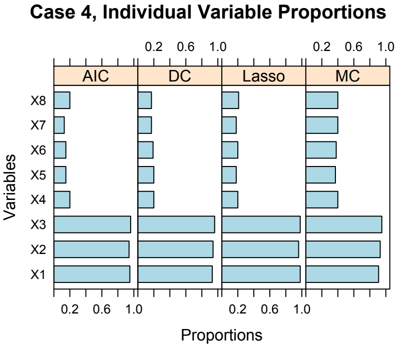
Case 3: Dependent but Uncorrelated Variables



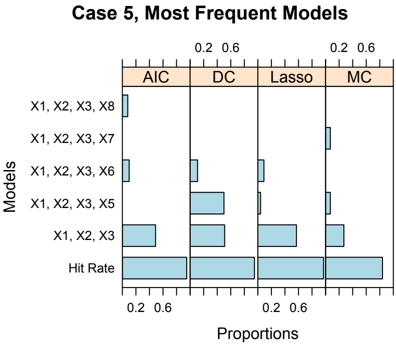


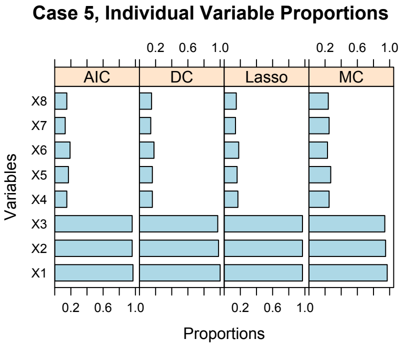
Case 4: Constant Collinearity Among Predictors



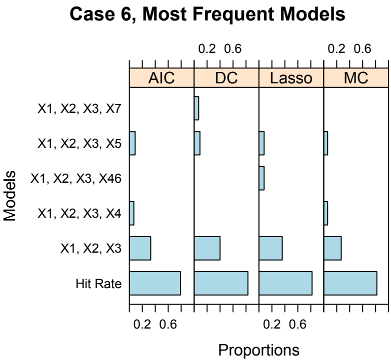


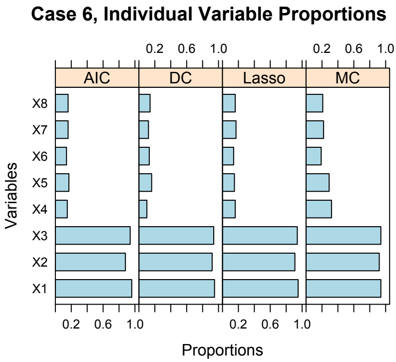
Case 5: Toeplitz Type Collinearity Among Predictors





Case 6: A Generalized Linear Model (Gamma Regression)



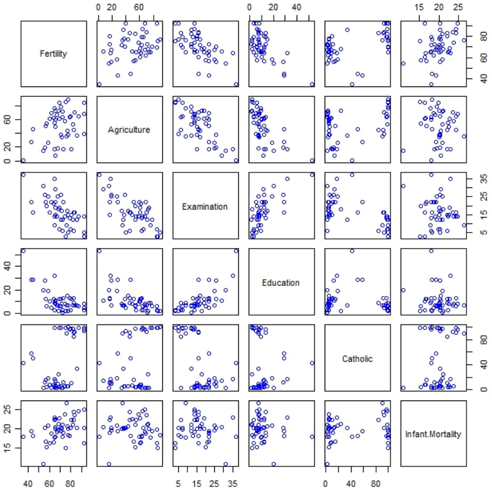


Application: S&P 500 Monthly Returns Data

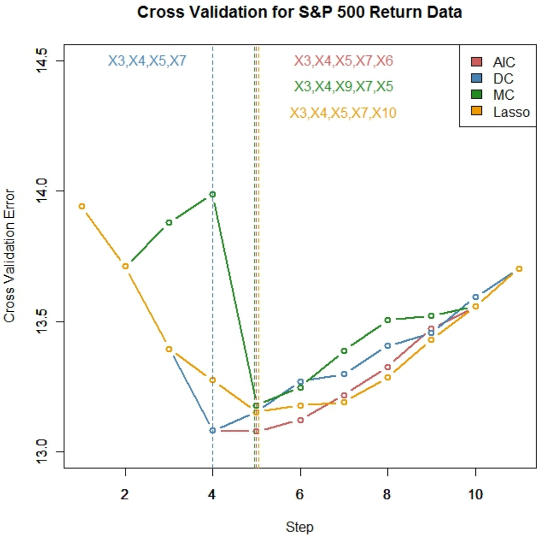
S&P 500 is an index portfolio defined by Standard & Poor's rating agency.

Monthly returns of S&P 500 index and the values of 11 candidate predictors between January 1989 and December 2007 (n=216) were analyzed using the four methods discussed above.

- Stepwise AIC
- Stepwise DC
- Stepwise MC
- Lasso


Fertility
Agriculture
Examination
Education
Catholic
Infant.Mortality

- Y - Monthly expected return of S&P 500 index (ex.r)
- X_1 - Dividend yield (div_yd)
- X_2 - Earnings yield (ern_yd)
- X_3 - Volatility index (vix)
- X_4 - Unexpected volatility (unvix)
- X_5 - Inflation rate (inf)
- X_6 - Change in inflation rate (inf_chg)
- X_7 - 90-day treasury bill (Tbill)
- X_8 - Industrial production index growth (ipi_gr)
- X_9 - Credit spread (cred_sp)
- X_{10} - Term spread (term_sp)
- X_{11} - Yield spread (yd_sp)


Cross Validation for S&P 500 Return Data
X3,X4,X5,X7
X3,X4,X5,X7,X6
X3,X4,X9,X7,X5
X3,X4,X5,X7,X10
AIC
DC
MC
Lasso
Cross Validation Error
14.5
14.0
13.5
13.0
2
4
6
8
10
Step

Conclusions

- Maximal Correlation and Distance Correlation were employed as comparison criteria in stepwise regression
- The methods are easy to implement
- The performances of the methods are comparable with commonly used methods
- In the presence of nonlinear or uncorrelated dependencies, our methods may be favorable

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